Duality as a Central Organizing Theme in Linear Circuit Analysis Instruction

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***Abstract*—(Full paper, Innovative Practice.) Linear circuit analysis is a complex topic in which students must use many principles to complete problems successfully, which may overload working memory and thereby impede learning. Introducing organizing principles may help students develop schemas that help reduce this burden and develop deeper conceptual understanding. The use of duality as such an organizing concept is explored in this work. To be effective, however, all the topics should be presented in a dual manner. Historically, definitions of series and parallel elements have been used that are not dual to each other, and mesh analysis has been performed in a way that is not fully dual to nodal analysis. This paper examines the research question of whether these key topics can be presented in a novel, fully dual fashion and whether students will accept and appreciate such a treatment. The revised approaches were implemented using lectures, online interactive tutorials, and step-based tutoring software exercises. Surveys using both quantitative and qualitative analysis were conducted over three semesters and showed positive reactions from 72-83% of students. These results can lead to development of a full set of instructional materials centered around duality to enable improved learning of circuit analysis.**

Keywords— Computer-aided instruction, duality, linear circuit analysis, mesh analysis.

I. Introduction

Student success in widely taught courses covering linear circuit analysis is crucially important given the ubiquitous presence of circuits in a vast range of engineered products and their continually increasing importance in measurement and control in many domains. Such courses are vital gateways for electrical engineers and are often required in many other engineering majors as well to support broad engineering competence. Circuits are however highly complex interconnected systems in which a change in any one element often affects the entire circuit. Moreover, students must master and utilize a bewildering array of ideas to solve problems, such as series and parallel connections, the difference between voltage and current sources (the latter often being unfamiliar from prior work), voltage and current dividers, short and open circuits, the very nature of current and voltage (the distinction often being poorly understood), inductors and capacitors, nodes and meshes and the associated analysis methods, Thévenin & Norton equivalent circuits, time domain and frequency domain, and so forth. It is no wonder therefore that many novices to this subject struggle to do so. Yet there is an underlying symmetry hidden in this complexity where all the above ideas and more can be systematically interrelated in pairs, simplifying the entire conceptual structure of the subject! This unifying idea, known as *duality*, was first expounded and exploited in detail by Russell in 1904 in his treatise on AC circuits [1], and has since been incorporated into many (but not all) textbooks on the subject [2-12]. Yet the prevailing approaches to certain key topics such as parallel and series connections and nodal and mesh analysis fail to recognize or exploit duality as discussed below, obscuring its significance in the process.

The present work is focused on developing and advocating new approaches to these two central topics that fully embrace duality and further improve fundamental definitions and procedures in the process. The central research question is whether a fully dual treatment of traditional topics can be developed and implemented in a way that will be embraced by students. Doing so will enable subsequent development of instructional materials in which duality plays a central role. Demands on working memory should therefore be reduced by limiting the number of distinct ideas employed at any one time, leading to improved learning based on cognitive load theory [13, 14]. Deeper and more transferable conceptual learning is expected to result, which is crucial in developing professional expertise [15].

II. Theoretical Framework

The basic framework used in this work is cognitive load theory (CLT) [13, 14]. Circuit analysis essentially consists of problem solving. Circuits often consist of many interconnected circuit elements, which are highly interrelated in that changes to one element typically affect the entire circuit. Solving such problems therefore involves high intrinsic load within CLT. Mental schemas in long-term memory are formed during instruction to understand the interrelationships among elements. However, many schemas may be required for a given problem and loading those into working memory may overload it in novice learners, resulting in inability to form and store new, more complex schemata to solve problems effectively. Experts differ from novices primarily in the number and complexity of schemas stored in long-term memory [13].

An important distinction is the level of structural knowledge [16] possessed by experts in long-term memory, which helps to translate declarative knowledge (such as

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"elements in series have the same current," "current is the flow of charge," "voltage is the potential energy per unit charge," and "resistance is voltage divided by current for a resistor") into procedural knowledge (such as "when I see two resistors in series, I can combine them into one resistor whose resistance is the sum of those of the original resistors." I.e., structural knowledge provides the knowing "why" relationships that link factual knowledge (knowing "that") to problem-solving procedures (knowing "how"). In this case, the structural knowledge might be that "voltages add for elements in series because a charge must change its energy each time it moves to a region with a different potential, so those changes are additive." Structural knowledge is used to organize and classify declarative knowledge schemas into larger "chunks" that can each occupy the limited number of "slots" in working memory, thereby minimizing cognitive overload. Structural knowledge also facilitates recall of information stored in long-term memory, and is a strong predictor of a student's ability to solve knowledge transfer problems [17]. It is used by experts to classify problems according to relevant principles, whereas novices tend to focus on surface features [18]. Congruence of structural knowledge between experts and novices is highly predictive of the problem solving abilities of the latter [19].

Based on the above principles, the approach of guided instruction is used here to minimize cognitive load and facilitate acquisition of schemata of structural knowledge consisting of high-level conceptual relationships in circuit analysis based on duality. Following recommendations in the literature [20], the basic principles are presented first, emphasizing duality, followed by worked examples to minimize cognitive load and finally by problem-solving in a step-based tutoring system.

III. Duality in Electric Circuits

Symmetry is one of the most pervasive and vital organizing principles in physics, mathematics, and related areas. Together with symmetry breaking, symmetry groups help explain the families of elementary particles, their interactions, and their properties such as mass, charge, and spin. Conservation laws are direct consequences of symmetry based on Noether's theorem [21]. Duality is a specific type of symmetry occurring in many domains, from ancient Chinese philosophy (yin and yang) to projective geometry to the Boolean algebra used to study digital circuits to wave-particle duality in quantum mechanics. In rigorous cases, it often involves a set of terms that can be exchanged in theorems to yield equally valid theorems, effectively doubling the knowledge in a field. In the projective geometry of a plane, for example, the words "line" and "point" can be so interchanged (e.g., "two points determine a line" implies that "two lines determine a point"). In Boolean algebra, the AND and OR operators can be exchanged (along with the literals 1 and 0) in any axiom or resulting theorem, so that theorems are normally presented in pairs in texts on this subject.

In circuit analysis, terms including those listed in Table I are dual to each other so that any theorem interchanging all of them remains valid [1], even if the circuit is nonlinear or time varying [5]. Another aspect of duality in this domain is that

TABLE I: Sets of Dual Terms in Circuits

Branch current	Branch voltage
Kirchhoff's current law (KCL)	Kirchhoff's voltage law (KVL)
Mesh (including outer mesh)	Node
Mesh current	Node voltage/potential
Loop	Cutset
Parallel	Series
Current source	Voltage source
Open circuit	Short circuit
Resistance	Conductance
Inductance	Capacitance
Sought Current[a]	Sought Voltage[a]
Sought Power[a]	Sought Power[a]
Control Current[b]	Control Voltage[b]

[a]Current, voltage, or power one wishes to determine
[b]For a dependent source

the geometric dual of any planar circuit obeys the exact same (possibly integrodifferential) equations as the original circuit if current and voltage variables are interchanged. To form an

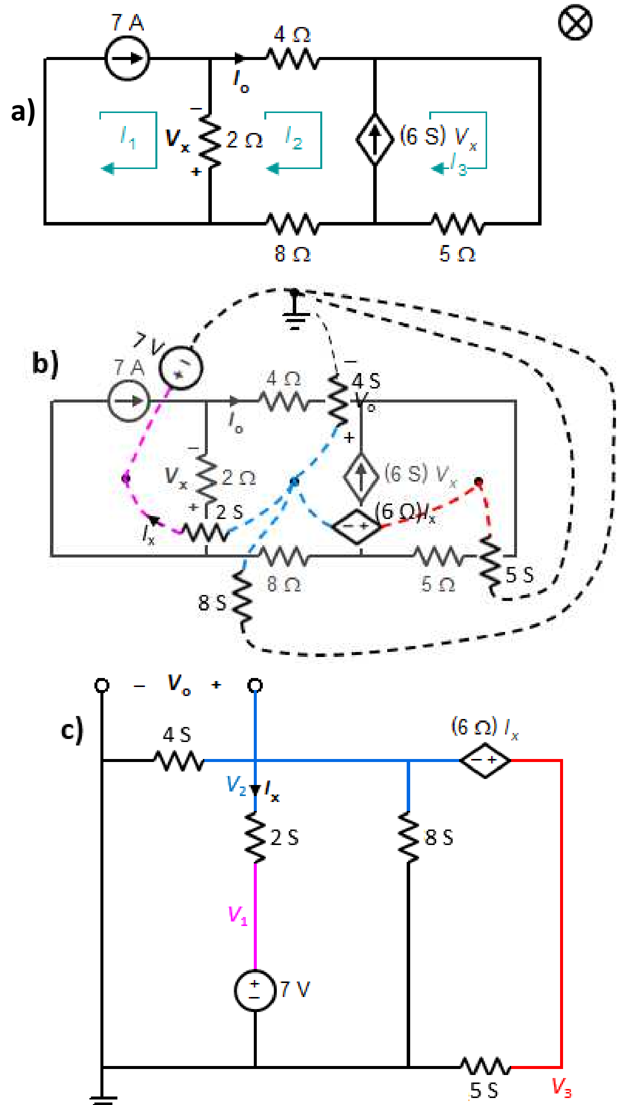

Fig. 1. a) A circuit whose exact geometric dual is to be constructed. (The symbol at upper right is a halt symbol denoting the reference mesh, as described below). b) A new node is placed in each interior mesh and outside the circuit (the latter is the reference node), and new elements are drawn across each existing element based on the entries in Table I, rotating source and voltage/current senses by +90°. c) The resulting dual circuit, re-drawn for clarity, with color-coded nodes.

exact geometric dual, similar to that in the theory of planar graphs [22], a node is placed within each mesh of a given circuit, including the region outside the circuit (called the outer mesh), as shown in Fig. 1. One then connects these nodes through the existing circuit elements, replacing each by a dual element with the same numerical value, based on Table I. Thus, a 2 Ω resistance is replaced by a 2 S conductance, a 7 V voltage source by a 7 A current source, and so forth. Sought variables (quantities one wishes to find) and control variables for dependent sources are similarly replaced, so that a current-controlled voltage source becomes a voltage-controlled current source, for example. The sense of each voltage source or voltage is defined as an arrow pointing from its negative side to its positive side and is in the direction of an arrow denoting a current source or current. The sense of each item with a polarity is rotated clockwise by 90° in the dual circuit (or alternatively, counterclockwise by the same amount). The direction is chosen to make the equations the same for the chosen type of analysis, depending on whether one chooses clockwise or counter-clockwise mesh currents and corresponding node voltages as V_1, V_2, etc. or $-V_1$, $-V_2$, etc.; both choices are arbitrary. The reference node and reference mesh (conventionally the outer mesh) must be connected to or contain the same elements in both cases, respectively.

As an example, the mesh equations for the circuit in Fig. 1a (using a supermesh consisting of meshes 2 and 3) are

$$I_3 - I_2 = (6\ \mathrm{S})\ V_x, \qquad I_1 = 7\ \mathrm{A}, \qquad I_o = I_2,$$

$$(I_2 - I_1)(2\ \Omega) + I_2(4\ \Omega) + I_3(5\ \Omega) + I_2(8\ \Omega) = 0, \text{ and}$$

$$V_x = (I_2 - I_1)(2\ \Omega).$$

The corresponding node equations for Fig. 1c (with a supernode consisting of nodes 2 and 3) are

$$V_3 - V_2 = (6\ \Omega)\ I_x, \qquad V_1 = 7\ \mathrm{V}, \qquad V_o = V_2,$$

$$(V_2 - V_1)(2\ \mathrm{S}) + V_2(4\ \mathrm{S}) + V_3(5\ \mathrm{S}) + V_2(8\ \mathrm{S}) = 0, \text{ and}$$

$$I_x = (V_2 - V_1)(2\ \mathrm{S}).$$

The corresponding equations are identical if currents (I) are changed to voltages (V) and vice versa (and units are interchanged). The node equations for Fig. 1a likewise correspond to the mesh equations for Fig. 1c.

The duality principle provides a framework to organize virtually all knowledge about circuits in a one-to-one correspondence that clarifies the structure of the entire subject.

IV. Defining Meshes Consistently

The above discussion highlights the need to redefine the term *mesh* in a circuit to support the duality principle. In virtually all modern textbooks and other sources, the term mesh is defined as a loop that does not enclose any smaller loops or elements in a circuit (having the appearance of a windowpane) [6-11, 23-27]. The periphery of the circuit (sometimes termed the outer mesh) is excluded in this definition. Yet, doing so causes an immediate problem in the duality transformations discussed above. To construct a geometric dual, a node is placed inside each mesh, but also necessarily in the outer mesh. This procedure suggests that the outer mesh should be treated on the same exact basis as all interior meshes. If it is not, the number of meshes in a circuit would differ from the number of nodes in its dual, which would prevent construction of dual circuits and substitution of dual terms.

In fact, if a planar circuit is drawn on the surface of a sphere using stereographic projection, which Whitney proved is always possible [28], the outer mesh in the plane drawing maps to a finite region of space surrounded by a loop of elements on the sphere, just like any other mesh. Given that circuits drawn in these two ways are electrically identical, there can be no logical basis to say that the number of meshes is different in the two cases. Further, the circuit on the sphere can be re-projected onto the plane in a way that makes *any* mesh become the outer mesh without changing any connections [28], so there is no reason to discriminate against the mesh that happens to lie on the outside for a particular way of drawing the circuit. One can even fold the circuit "inside out" on a plane to achieve the same transformation [29].

An improved definition is therefore that

A mesh is a loop that does not enclose any smaller loops, or that is not enclosed by or a portion of any larger loop in a planar circuit.

The outer mesh is thus placed on an equal footing with interior meshes. This approach is consistent with planar graph theory, where the term *cycle* corresponds to a mesh in a circuit, and the term *face* denotes the region either enclosed by a cycle, or outside the graph (called the *infinite face*) [22]. In graph theory it is widely accepted that the infinite face is equivalent to any other face [22]. Moreover, Guillemin argued that the term mesh itself should refer to the regions enclosed by (or outside, for the outer mesh) an elementary loop through circuit elements [3]. In this view, as in our definition, even the simplest single loop circuit has two meshes, the inside and outside of the loop. Its dual is a single node-pair circuit, so it is logical that it should be a single mesh-pair circuit. The circuit of Fig. 1a then has four meshes, not three as usually assumed.

V. Dualizing Mesh Analysis

Recognizing that a circuit always has one more mesh than the number it is conventionally said to possess can help resolve a long-standing mystery, namely why the conventional textbook procedure for mesh analysis is *not* fully dual to that for nodal analysis. Specifically, the first step in nodal analysis is always to select a reference node whose voltage is conventionally set to zero, and then number only the remaining nodes [6-11, 23-27, 30-32]. Yet no such step conventionally appears in mesh analysis, even though it should be the exact dual of nodal analysis. The very first detailed exposition of mesh analysis by Fleming [33], however, based on notes of lectures by James Clerk Maxwell, the inventor of mesh analysis, contains a clue to this discrepancy. He states that the current of the outer mesh (called the "cyclic symbol of exterior space" in that paper) is taken to be zero. Yet this assumption seems to have been overlooked in most subsequent work, with the exception of a few more advanced treatments [3, 5, 34]. For consistency (e.g., when the circuit is drawn on the surface of a sphere), the

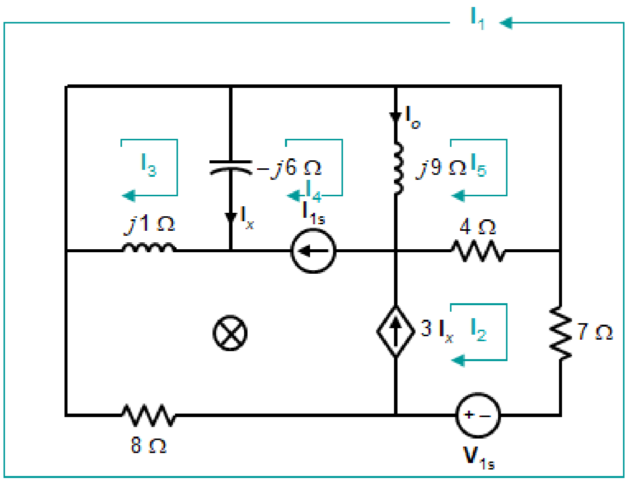

Fig. 2. Example of an AC phasor-domain circuit where an interior mesh has been selected as the reference mesh using a halt symbol, to simplify the resulting analysis. The counterclockwise outer mesh current is shown. Meshes 2 and 4 become part of the reference supermesh so that no KVL equation need be written for them, and no supermesh need be explicitly defined. The label $\mathbf{I}_{1s}$ in the figure denotes $1\angle 120°$ A, and $\mathbf{V}_{1s}$ is $3\angle 0°$ V. The symbol $\mathbf{I}_o$ denotes a "sought current;" i.e., one we wish to find.

current of the outer mesh should be directed counterclockwise if the interior mesh currents point clockwise, as shown in Fig. 2. With this more complete view of mesh currents, it becomes clear that *every* branch current is in fact the difference of two mesh currents, not just those of interior elements. It is simply that one of the two can be zero. This situation now mirrors that of branch voltages, which are *always* the difference of two node voltages, one of which might be zero.

A consequence of this viewpoint is that no mesh current is ever absolute in nature or even measurable; only *differences* of mesh currents can be measured. Duality in fact demands that if such a statement is valid for node voltages, as is generally accepted, it must be equally valid for mesh currents. Yet at least one existing handbook claims that exterior mesh currents are in fact absolute and measurable [35], which is now seen to be a fallacy. Both node voltages and mesh currents are purely *fictitious* quantities defined to obviate the explicit application of Kirchhoff's voltage law (KVL) in nodal analysis and of Kirchhoff's current law (KCL) in mesh analysis, respectively. Adding any constant value to all node voltages leaves all branch voltages unchanged, just as adding any constant to all mesh currents leaves all branch currents unchanged. A different choice of reference node merely adds a constant to all node voltages, with no consequence for any measurable quantity (like a gauge transformation in electromagnetics).

A logical extension of this analogy follows from the fact that one is free to choose a reference node to advantage in nodal analysis. Typically, selecting one that is connected to voltage sources can reduce or eliminate the need to use supernodes, which have less regular KCL equations than do normal nodes. Further, a reference node connected to many circuit elements is typically somewhat advantageous in that the corresponding complicated KCL equation is conventionally not written. By duality, it should be possible to exercise similar freedom by actively choosing a reference mesh to be one that contains current sources and that contains many elements. One could do so by re-drawing the circuit with the desired reference mesh as the outer reference mesh, but such a tedious and difficult process is not needed. There is no reason that an inner mesh cannot be selected as a reference mesh, if the outer mesh current is then maintained instead of being suppressed as is conventionally done. The reference node is normally designated by a ground (datum node) symbol being attached to it. We therefore propose a corresponding symbol to designate an arbitrary reference mesh, called a *halt symbol*. It consists of a circle enclosing an X, as shown in Fig. 2, and is placed within a mesh (or anywhere outside the circuit to designate the outer mesh as the datum mesh). The mesh current of the reference mesh is defined to be zero, so only the remaining mesh currents need be drawn and numbered. When a different reference mesh is (arbitrarily) selected, all mesh currents are shifted by the same amount (the current of the new reference mesh when the old reference was used). The purely fictitious nature of mesh currents is thus exposed. This behavior is now exactly dual to the effect of choosing a different reference node, as it should be.

Using this more flexible approach makes the mesh analysis procedure completely dual to the nodal analysis procedure, as it should be to support duality. Interior and exterior current sources are now both treated exactly the same. In a circuit like that in Fig. 2, it is advantageous to pick an inner reference mesh instead an outer one, simplifying the resulting equations. For example, the mesh equations with the reference mesh chosen in Fig. 2 are

$$\mathbf{I}_4 = 1\angle 120°\text{ A},\qquad \mathbf{I}_2 = 3\mathbf{I}_x,\qquad \mathbf{I}_o = \mathbf{I}_4 - \mathbf{I}_5,$$

$$\mathbf{I}_x = \mathbf{I}_3 - \mathbf{I}_4,\qquad \mathbf{I}_3(j1\ \Omega) + (\mathbf{I}_3 - \mathbf{I}_4)(-j6\ \Omega) = 0,$$

$$(\mathbf{I}_5 - \mathbf{I}_4)(j9\ \Omega) + (\mathbf{I}_5 - \mathbf{I}_2)(4\ \Omega) = 0,\text{ and}$$

$$\mathbf{I}_1(8\ \Omega) + 3\angle 0°\text{ V} + (\mathbf{I}_1 - \mathbf{I}_2)(7\ \Omega) = 0.$$

(The last equation is the KVL equation for the outer mesh, summing voltage drops in the direction of the outer mesh current.) If the outer mesh were instead chosen as the reference mesh and the interior ones were numbered the same except that the lower left mesh now has clockwise current $\mathbf{I}_1$, the equations would instead be

$$\mathbf{I}_4 - \mathbf{I}_1 = 1\angle 120°\text{ A},\qquad \mathbf{I}_2 - \mathbf{I}_1 = 3\mathbf{I}_x,\qquad \mathbf{I}_o = \mathbf{I}_4 - \mathbf{I}_5,$$

$$\mathbf{I}_x = \mathbf{I}_3 - \mathbf{I}_4,\qquad (\mathbf{I}_3 - \mathbf{I}_1)(j1\ \Omega) + (\mathbf{I}_3 - \mathbf{I}_4)(-j6\ \Omega) = 0,$$

$$(\mathbf{I}_5 - \mathbf{I}_4)(j9\ \Omega) + (\mathbf{I}_5 - \mathbf{I}_2)(4\ \Omega) = 0,\text{ and}$$

$$(\mathbf{I}_1 - \mathbf{I}_3)(j1\ \Omega) + (\mathbf{I}_4 - \mathbf{I}_3)(-j6\ \Omega) + (\mathbf{I}_4 - \mathbf{I}_5)(j9\ \Omega) + (\mathbf{I}_2 - \mathbf{I}_5)(4\ \Omega) + \mathbf{I}_2(7\ \Omega) - 3\angle 0°\text{ V} + \mathbf{I}_1(8\ \Omega) = 0,$$

which are clearly more complex. (The last equation is KVL for a supermesh consisting of meshes 1, 2, and 4.)

VI. Redefining Series Connections

Another aspect of conventional circuit analysis that fails to obey duality is that of series and parallel connections (the latter being originally being referred to as "elements forming multiple arcs," which avoided confusion with the geometrical meaning of "parallel" [33]). The usual definition of parallel elements is those having the same physical voltage, therefore being connected to the same pair of nodes [7-11, 23-26, 30-32]. This definition is simple, elegant, easy to apply, and fully inclusive. Yet the usual definition of series elements is *not* the dual of the above statement. Instead, it is typically stated that elements are in series if they have the same physical current, due to being pairwise connected to a common node, with no other conducting path connected to that node [6-8, 10, 11, 23, 24, 27, 30-32]. Further, a transitivity property is implied, so that if A is in series with B, and B is in series with C, then A must be in series with C. This condition is certainly sufficient for elements to be in series, but by no means necessary. For example, the 4 Ω and 8 Ω resistors in Fig. 1a have no nodes in common at all, and neither is in series with any other element. Yet they must have the same magnitude of current, as easily proved by applying KCL to the subcircuit (one-port) consisting of a current source in parallel with a resistor on either end of the circuit. They could be combined to form a single 12 Ω resistor, replacing one of them by a short.

The usual end-to-end definition could be salvaged by generalizing it to include both subcircuits and individual elements, to cover cases like that in Fig. 1a. Doing so algorithmically can be very complicated for general circuits, however, as subcircuits can be enclosed in larger subcircuits and can overlap each other, making it difficult to identify the specific subcircuits that form a series connection with individual elements. A vastly better approach is to start with the normal definition of parallel and form its dual. Namely,

Two or more elements are in series if they are connected in the same pair of meshes.

(For non-planar circuits, this can be modified to have elements connected in the same set of fundamental loops.) For this definition to work, one must include the outer mesh as a true mesh as was done above, which is logically necessary for other reasons anyway. In Fig. 1a, for example, both the 4 Ω and 8 Ω resistors are part of the central mesh and outer mesh. Being part of the same two meshes obviously implies that their branch currents, being the differences of the two relevant mesh currents in each case, must always be the same. This definition is simple, elegant, and easy to apply without any need for subcircuits, and needs no additional transitivity property. It is easily implemented as an algorithm by creating a *meshlist* for a circuit, which lists each element and the two meshes in which it is connected, and then comparing the entries in the list. Such a list is analogous to the normal netlist (or nodelist) used for example in SPICE analysis.

The usual chain-based definition can still be used when desired, but the new approach is more general and gives a complete list of elements in series, which the usual one does not always do unless subcircuits are included (with great algorithmic difficulty). It requires introducing the idea of meshes a bit earlier than usual, but students may benefit from that early introduction when treating mesh analysis later. A major advantage of the new definition is its fully dual relationship with the usual definition of parallel elements. The dual of the traditional definition of series elements would be that elements are in parallel if they are both in the same mesh and no other element is contained in that mesh. Yet this definition clearly fails for the 8 S and 4 S resistors in Fig 1c, even though those elements are obviously in parallel, just as the usual series definition fails for its dual circuit in Fig. 1a. Use of the node-based definition of parallel elements and its dual mesh-based definition of series elements is clearly superior. Nodes are not the best tool to identify series connections, because they are linked to voltages rather than to currents.

VII. Implementing the New Approaches in Instruction

Lecture notes used by the first author for mesh analysis were revised to explain the outer mesh and its importance and to include selection of the most advantageous reference mesh using the halt symbol starting in Fall 2019 and continuing through Spring 2021 (PowerPoint was not used and the material was written on a blackboard instead or on a tablet when teaching remotely in Fall 2020-Spring 2021). Figures showing how a given circuit can be redrawn with any mesh as the outer mesh [29] and pictures of that circuit drawn on the surface of a sphere were used (together with rubber balls on which the circuit had been drawn that were passed around in live lectures). The procedure for mesh analysis was revised to be exactly dual to that for nodal analysis. The first author's lecture explanations of series relationships were revised in Spring 2020 to show them in terms of elements having the same pair of mesh currents.

Further, an introductory interactive multiple-choice tutorial was created on mesh analysis in Fall 2019 and incorporated into the existing step-based tutoring system called Circuit Tutor [36-39]. This tutorial introduced the new definition of meshes that includes the outer mesh as a mesh, and the ideas of a selectable reference mesh and halt symbol as discussed above. It covers the entire process of mesh analysis including current constraint equations for current sources, supermeshes for current sources not contained in the reference mesh (instead of exterior current sources in the old approach), and dependent sources. A previously existing introductory tutorial on series and parallel connections was revised in late Spring 2020 to include these new features of meshes and to explain series connections using both the traditional definition with subcircuits and the new definition using mesh currents.

The introductory tutorial on both topics is followed by step-based tutoring instruction using circuits whose topologies and element values are both generated in a random fashion as described previously [36-38]. The "game" in each case includes both fully worked and explained examples at four progressive levels of difficulty and corresponding exercises. The worked examples are included in line with our theoretical framework as they are known to be an effective way to minimize excessive cognitive load [13, 20]. The series-parallel game presents random circuit diagrams and asks students to click on sets of elements that are either in series or in parallel, followed by clicking a button to check the appropriate case. Checkmarks are placed directly on the elements when they are selected to minimize extraneous cognitive load. Immediate feedback is provided with the option of detailed explanations of wrong answers. Students may also "give up" on a problem at any time for no penalty (grading is based purely on completion) to see a full explanation and solution, followed by a new problem of the same type (or by viewing additional worked examples first). Explanations and visualizations were introduced starting in late Spring 2020 on the mesh-based definition of series elements, including display of color-coded mesh currents and colored highlighting of subcircuits that make individual elements be in series using the traditional chain-based definition. A YouTube video is also available from within the software to illustrate both the operation of the user interface and to show example problems being worked at each level. This approach can help to minimize cognitive overload by using both auditory and visual portions of working memory, as the problem solving is narrated [13]. A complete PDF transcript of each student's work is generated and stored (including both correct and incorrect answers, so labeled) for their later use while studying or reviewing. The series-parallel game incorporates a built-in pre-test and post-test (two problems each), but such tests have not yet been implemented for mesh analysis.

For the mesh analysis topic, four separate "games" are available on writing the relevant equations (without solving them) in both DC and AC (phasor) cases, and on solving the complete problems (when given the KVL equations) for both DC and AC circuits. Students select the type of equation to enter (current constraint due to a current source, KVL for a mesh or supermesh, equations for circuit variables controlling dependent sources, and sought variable equations for any specified branch currents, voltages, or powers (the latter for now only in DC circuits) [36-38]. The placement of the halt symbol in an inner mesh is not yet supported but is planned to be. Once an equation type is selected, the student is now asked (since Fall 2020) for which mesh(es) they wish to write a KVL equation or for which source they wish to write another type of equation. If they select inappropriate meshes, they are immediately warned and charged with an error (a certain number of errors causes a game to be forfeited and to have to be repeated for credit).

Students are then presented with a palette of terms of appropriate types for the selected type of equation, which they then drag & drop into an equation builder and fill in the blanks to create an equation. Once entered, they are given immediate feedback on its correctness and potentially charged with an error and given a chance to correct it if they have not exceeded their allowable errors. In the games requiring complete solutions, they must re-formulate their equations into standard form, then create and solve the appropriate matrix equation (doing the algebra on paper), and finally compute the desired sought quantities using a calculator (with complex number abilities for AC problems). Each step is immediately checked, and errors are potentially charged. They can give up at any point to see a fully worked and explained solution without penalty (but must then complete another full problem of that type).

VIII. Data and Results

An initial analysis of the new version (2.0) of the series-parallel game was conducted in Spring 2020 with students who had already completed the prior version 1.0 at the beginning of that semester (which used the traditional chain-based definition and would not even accept sets such as the 4 Ω and 8 Ω resistors in Fig. 1a as a valid series set). These students in three different sections (with two instructors) were offered extra credit to complete the revised introductory tutorial and game (v. 2.0) and to complete a survey comparing the two. A total of 72% of 88 students either strongly or somewhat agreed that the mesh-based definition of series connections is better than the chain-based definition they learned originally, and the same percentage strongly or somewhat agreed that future students should use version 2.0 in preference to the prior version. (These percentages were 80 and 87%, respectively, in the section of an instructor who used the new definition in lecture as well.) Only 11% of students strongly or somewhat disagreed that the new definition is better, and 13% strongly or somewhat disagreed that future students should use version 2.0. The remainder in each case were neutral.

All students used v. 2.0 of the series-parallel game in Fall 2020 and were surveyed on their opinions of the new approach at the end of the semester. A total of 80 students responded, mainly from four class sections at two different institutions. Of these, 75% felt that the new definition was somewhat or much better than the old one, 16% were neutral, and only 9% felt it was somewhat or much worse. When recommending which definition students should use in the future, 29% favored using only the mesh-based definition, 10% favored using only the traditional definition, and 51% favored using a combination of the two as is now done in Circuit Tutor (the remainder felt it did not matter). Analysis of qualitative comments in Spring 2020 by dividing them into (non-mutually exclusive) categories showed the most common views to be that the new method yielded more complete understanding and identified series elements missed by the old method, and that the new method is better than the old one for a variety of other reasons (29 students each). A total of 19 students preferred the old method, feeling the new one was too complicated or confusing. Comments in Fall 2020 were similar. Four students mentioned favoring the dual nature of the new approach.

The new version of the introductory tutorial on mesh analysis and the related game were used in 57 different class sections taught by 24 different instructors from Fall 2019 through Spring 2021 at 7 different institutions of different types (including 3 minority-serving institutions). Over 1800 students used them. Between 90-95% of these students rated the activity as "very useful" or "somewhat useful" to learn mesh analysis (as opposed to "not very useful" or "a waste of time") in a survey administered automatically at the end of each game. To assess student opinions of the new approach, a total of 10 sections at four institutions were given a survey after using the material in Fall 2019, Fall 2020, and Spring 2021. Of the 195 respondents, 42% felt the new approach was much better than the traditional one in their textbooks, 41% felt it was somewhat better, 11% felt it was about the same, and 9% felt it was somewhat or much worse. For hypothesis testing using a 50% threshold for positive reactions from students to the new approach compared to the traditional approach, a one-sample *t*-test found that significantly more students had positive reactions to the approach than anticipated ($p < .01$). Further, 70% of respondents felt that students who had not yet studied this topic should learn it with the new approach, 25% felt it did not matter, and only 5% recommended the traditional approach. Qualitative comments were also solicited in Fall 2019 and Spring 2021. Of the 63 comments received, 38 were considered positive towards the new approach, 7 as neutral, 5 as negative, and 13 as irrelevant or no opinion. Six of the comments specifically mentioned the duality and symmetry between nodal and mesh analysis as helpful for learning. Some example comments are shown in Table II.

A special effort was made to emphasize the symmetry between nodal and mesh analysis and to improve student learning by incorporating the "desirable learning difficulties [40]" of spacing student work over time and interleaving the two topics of nodal and mesh analysis in two sections taught by the first author in Spring 2021. Students in each section were randomly assigned to one of two groups, A and B. Group B had a single due date for the four games involving either DC nodal or mesh analysis, which they could complete in any order. Group A had four different due dates spread over the space of a week. The first level of the node equations game, node solutions game, mesh equations game, and mesh solutions game were due after two days; the second levels of each were due two days later; the third levels two days later; and the fourth levels three days later (the same day that all the work was due for Group B). Students in Group A were thereby encouraged to compare and contrast the nodal and mesh analysis methods by being required to "interleave" the two topics in their work. For AC nodal and mesh analysis later in the semester, the roles of Groups A and B were reversed for fairness. Learning was assessed (to a limited degree) using a single exam problem on either nodal or mesh equations (depending on section) that formed 20% of the Hour Exam #1 grade for the DC topics and 22% of the Hour Exam #2 grade for the AC ones. Hypothesis testing using an independent *t*-test analysis found no statistically significant differences on the scores for those problems.

TABLE II. SAMPLE STUDENT COMMENTS ON THE NEW APPROACH

- The duality concept is genius. I think it is very important for students to understand and grasp the concept, even if it's a bit confusing (Which it isn't...)
- The new approach really helped me with mesh analysis because, although it confused me at first, it made things a lot easier once I got the hang of it. Especially since it was similar to the way we dealt with node analysis. I liked it.
- I never learned mesh analysis by textbook or the traditional method. However, getting to look at both nodal and mesh analysis through the same lens made it very easy to catch on. Since the same line of logic applies to both, it didn't feel like I was learning two separate topics, but rather two different applications of the same principles, which made a lot of sense to me.
- Tying mesh and nodal analysis together helped my comprehension and made retention easier
- The difference between working with currents and voltage is flipped. It is the same, but mirrored. I love that this new approach helped complete the symmetry of the two approaches.
- I've taken this Circuits 1 online before and many mistakes with the old method. I've been more successful by using the new method.
- The new approach is definitely beneficial for displaying the concept of duality. However, it does not necessarily make mesh analysis easier than if it was ommitted entirely. With that being said, it also does not make it harder.
- Having a reference mesh was a bit confusing to me.
- It's useful in cases in which there are current sources within the circuit but not on the outer mesh, which can make the problems easier to solve in avoiding using supermeshes
- The more similar the steps are to nodal analysis, the harder they will be to mess up. Teaching mesh analysis in this way has been extremely beneficial due to how similar they look through the techniques learned in class.
- As long as practice opportunities are provided on how to determine where to put the reference mesh, the new method will work well. The lack of practice discouraged me from picking a reference mesh that wasn't the outer loop due to my worries of if I am abiding by passive sign convention or not.

A survey was administered on the above experiment, which asked students in one question if they felt that they were able to get a deeper understanding of nodal and mesh analysis by comparing and contrasting those methods. Of the 34 respondents, 74% either strongly agreed or agreed, 9% disagreed, none strongly disagreed, and 12% were neutral. In exam problems involving DC or AC mesh analysis, 50%, 45%, and 61% explicitly marked a reference mesh in three separate cases. In two exam problems where it was not particularly advantageous to pick an inner mesh as the reference mesh, 0% and 8% of students did so, and in one problem where it was advantageous, 13% did so. Higher rates of usage will likely be achieved if Circuit Tutor is modified to allow choosing a mesh other than the outer mesh as a reference mesh, to allow homework practice using that approach.

IX. Analysis and Discussion

Only two major topics usually taught in linear circuit analysis courses are typically treated in ways that do not fully comport with duality, namely the non-dual traditional definitions of series and parallel connections and mesh analysis. The above discussion has affirmed that both topics can be approached from fully dual points of view, which moreover have other advantages (e.g., simplifying mesh equations in some cases by an optimal choice of reference mesh, and identifying series elements that would be missed using the conventional definition). A positive answer can therefore be given to the first portion of the research question, whether these topics can be presented in a fully dual fashion. The second portion of the question asks if students will accept and appreciate these new treatments. The survey data discussed above show that students felt the new approaches were better than the old ones by margins of 72%, 75%, and 83% in different surveys, and only 11%, 11%, and 9% thought the new approaches were worse. Regarding future use, 72% favored the revised approach (including both mesh-based and connection-based definitions of series elements) and only 13% favored the prior version of series and parallel instruction. In a second survey, 79% favored using the mesh-based definition either exclusively or in combination with the traditional one, and only 10% favored using only the traditional one. For mesh analysis, 70% recommended the new approach and only 5% specifically recommend the traditional one. Taken together, these surveys clearly show that students not only accept approaches that embrace duality but actually prefer them, answering the second part of the research question in the affirmative.

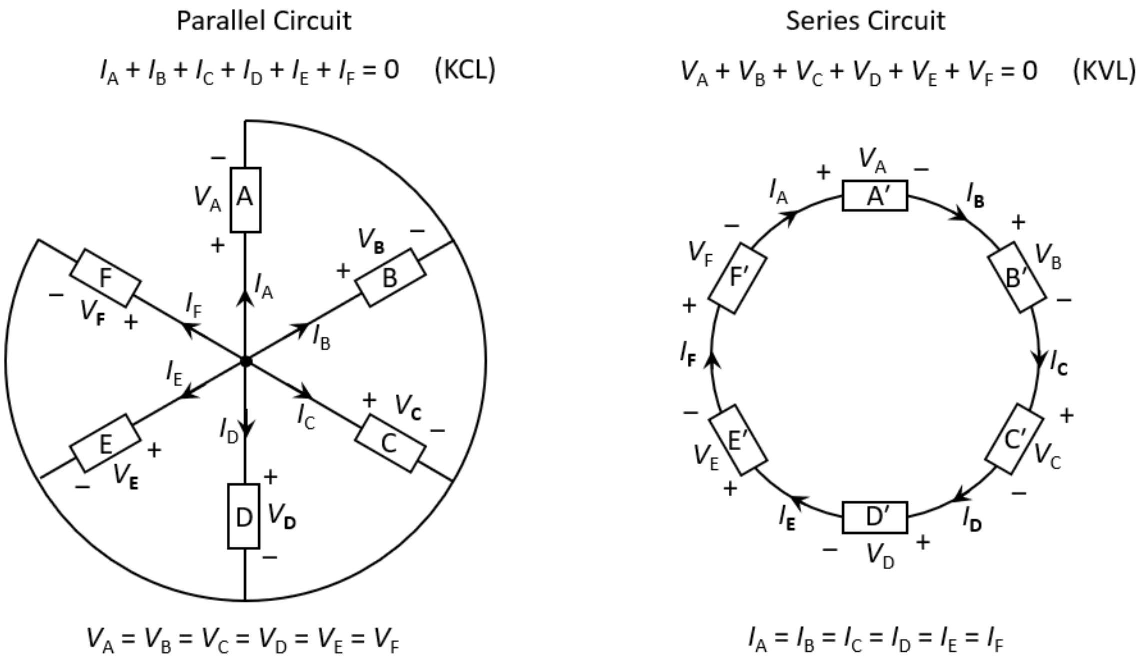

Fig. 3. Example of a graphic organizer illustrating basic aspects of duality, including the dualism between KCL and KVL, between series & parallel circuits, the duality of nodes and meshes, etc. Various similar diagrams could be used to present basic ideas of circuit analysis in a dual, side-by-side format. (The boxes labeled A-F are arbitrary circuit elements, and those labeled A′-F′ are their duals based on Table I. One arc of the wheel at left must be omitted to avoid redundant shorts, whose dual would include an isolated node on the right.)

Having reformulated the key obstacles to using duality, the next step will be to develop a full set of instructional materials that develop dual topics in a one-to-one correspondence with their duals, such as KVL and KCL, voltage division and current division, short and open circuits, series and parallel RLC circuits, etc. It is expected that doing so could result in improved understanding of the distinctions between current and voltage, which are often confused by students beginning (or even completing) courses in circuit analysis [41, 42]. These materials should incorporate simple graphic organizers to help students understand, as doing so is known to be effective [20]. A possible example is shown in Fig. 3. Once such materials are fully developed, the next research step should be to determine if a dual approach can improve learning, retention, and transfer.

X. Conclusions

Two key subject areas in linear circuit analysis have been shown to be inconsistent with the underlying duality of electric circuits, namely the conventional definition of series elements and the usual approach to mesh analysis. To be fully inclusive and dual to the normal definition of parallel elements, series elements should be defined as those belonging to the same pair of meshes (and therefore having the same currents), just as parallel elements are those connected to the same pair of nodes (therefore having the same voltages). Mesh analysis needs to permit the flexibility of explicitly choosing a reference mesh anywhere in the circuit, just as nodal analysis does. In both cases the basic variables (node voltages and mesh currents) should be recognized as being equally fictitious and unmeasurable. For both of the above areas, it is essential to recognize that the outer mesh of a circuit is in every way equivalent to inner meshes, so that the very definition of mesh must be revised to comport with duality.

These ideas have been successfully incorporated into traditional lectures as well as a step-based learning system, and student surveys indicate strong (72-83%) endorsement of the revised approaches. Further work should center on developing learning materials that use duality as a central theme to help create structural knowledge. That knowledge can be used by students to form expert-like schemas while avoiding excessive load on their working memory. The duality concept can be further be exploited when comparing the time and frequency domains in phasor, Laplace, and Fourier-based circuit analysis.

XI. Acknowledgment

The authors wish to thank the 22 other instructors who have used the materials in Circuit Tutor to teach the concepts in this paper to their students. Circuit Tutor is available completely free to any instructor who wishes to adopt it and their students; contact skromme@asu.edu for an access code.

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